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A Review of Kraichnan's Theory of Turbulence

MAY 1967

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Plasma Research Laboratory
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Prepared for BALLISTIC SYSTEMS AND SPACE SYSTEMS DIVISIONS
AIR FORCE SYSTEMS COMMAND
LOS ANGELES AIR FORCE STATION
Los Angeles, California

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FOREWORD

This report is published by the Aerospace Corporation, El Segundo, California, under Air Force Contract AF 04(695)-1001.

This report, which documents research carried out from September 1964 through September 1966, was submitted on 21 April 1967 to Captain Ronald J. Starbuck, SSTRT, for review and approval.

Approved

R. X. Meyer, Director
Plasma Research Laboratory

Publication of this report does not constitute Air Force approval of the report's findings or conclusions. It is published only for the exchange and stimulation of ideas.

Ronald J. Starbuck
Captain, United States Air Force
Chief, Space Environment and
Electronics Branch

ABSTRACT

The principal assumptions used by R. Kraichnan in formulating a theory of turbulence are reviewed, and their meaning is examined in the light of a mathematical model. A regression function is defined and used to evaluate the triple correlations. The importance of the fourth order cumulants is discussed. The merits of the theory are illustrated by comparison with experimental results--in the case of the mathematical model -- of grid turbulence (skewness factor) and of turbulence at very large Reynolds number.

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FIGURES

I. INTRODUCTION

The object of this review is to examine the general method proposed by R. Kraichnan several years ago. We will probe the nature of the assumptions. The numerous mathematical details will receive little attention. The theory was first proposed in the context of MHD turbulence, where comparison with experiment was not possible. The problem was then reduced to the case of ordinary turbulence (homogeneous, isotropic, and incompressible), and some agreement was found.

There is little doubt that Kraichnan's general approach can be extended to MHD problems, to turbulent shear flows, or to plasma turbulence. In fact, a whole category of problems of physics may benefit from any success in the field of nonlinear random problems. For example the method may apply to nuclear magnetic resonance.

The method of Kraichnan is characterized by the use of a regression function that describes the average effect of a small perturbation imposed on the turbulent flow. In order to introduce this function as clearly as possible and to show that it can be measured, I shall refer to a model problem. This problem is only a mathematical exercise, void of any physical reality, but easier to treat than the simplest case of turbulence. The model problem has the advantage that it can be integrated numerically with a large computer so that the theory can be compared to experimental results.

II. STATEMENT OF THE PROBLEM

Let us start from the equations of Navier-Stokes written in the following, somewhat informal form:

$$(\partial v_\bullet/\partial t) + \Sigma\, v_\bullet(\partial v_\bullet/\partial x_\bullet) + (\partial p/\partial x_\bullet) = \nu(\partial^2 v_\bullet/\partial x_\bullet \partial x_\bullet)$$

$$\Sigma\,(\partial v_\bullet/\partial x_\bullet) = 0. \tag{1}$$

where $v(x,t)$ is the Eulerian velocity and p the pressure. The density is unity, ν is the viscosity, and the dots represent assorted indices 1, 2, 3. The summation extends over three terms, and there is a total of four equations.

In order to obtain a system of ordinary nonlinear differential equations, instead of partial differential equations, we shall now use a Fourier transformation in the space coordinates. The Fourier transforms $u(k,t)$ are defined such that

$$v = \Sigma\, u_\bullet(k,t)\exp(ik_\bullet x_\bullet) \tag{2}$$

The symbol k represents a wave vector, and the boundary conditions correspond to a periodic flow. The interesting part of the flow is confined to a cubic box of size L; therefore the lowest magnitude of the wave vector shall be $2\pi/L$. As we consider higher modes, the magnitude of k increases in steps, and many different orientations are possible. In principle, the magnitude of k could go to infinity, but we do not want to deal with wave lengths comparable to microscopic scales, such as the mean free path. Thus we shall specify that the magnitude of k has a maximum. As long as this maximum is high enough, say ten times above the Kolmogoroff threshold, it has no significance. The summation of Eq. (2) therefore covers a large but finite number of terms.

By entering Eq. (2) into Eq. (1) and collecting the terms proportional to e^{ikx}, we obtain the equations of Navier-Stokes in the following form:

$$\partial v_i(k)/\partial t = \Sigma\, a_{ijh}\, u_j(k')\, u_h(k'') - \nu k^2 u_i \qquad (3)$$

The coefficients a_{ijh} are exactly defined from the Navier-Stokes equations. The two Fourier components multiplied by a_{ijh} correspond to two wavenumbers k' and k" such that k' + k" = k. Thus the summation is extended over all triangles in a wavenumber space. Perhaps one could formally include all pairs of vectors and decide that a_{ijh} is zero unless the corresponding vectors form a closed triangle.

For each magnitude and orientation of k there are three such equations, so that we have now a very large system of nonlinear ordinary differential equations. The continuity equation is implicit.

I will now recast this system in a form which is more appropriate for a review. The new form will also be more general and will deal only with real quantities. Indeed, every $u_i(k,t)$ and every a_{ijh} is complex. Beginning with the lowest magnitude of k and a first orientation, we define

$$u_1(k_0,t) = F_1(t) + iF_2(t)$$

$$u_2(k_0,t) = F_3(t) + iF_4(t) \tag{4}$$

$$u_3(k_0,t) = F_5(t) + iF_6(t)$$

Proceeding to the next orientation or magnitude of k, we write

$$u_1(k_1,t) = F_7(t) + iF_8(t)$$

$$u_2(k_1,t) = F_9(t) + iF_{10}(t) \tag{5}$$

$$u_3(k_1,t) = F_{11}(t) + iF_{12}(t)$$

and so on down the list of wavenumbers. The last expression is

$$u_3(k_{max},t) = \ldots\ldots\ldots + iF_N \tag{6}$$

This defines the real quantities F_1 to F_N where N is a very large number. For the simplest case of turbulence N would be of the order of 10^6. Each F_i varies with the time. Note that there are six F's for each wavenumber, so that the large wavenumbers correspond to functions of large indices.

The equations of Navier and Stokes can be written in terms of the new functions as follows:

$$dF_i/dt = \Sigma\, c_{ijk}F_jF_k + \Sigma\, b_{ij}F_j \tag{7}$$

where the coefficients c_{ijk} are purely real and related to the coefficients a_{ijk} of the previous formulation. The coefficients b_{ij} come from the viscous terms. These equations are of a very general form, and they encompass a wide assortment of problems. Shear flow turbulence, MHD turbulence, or plasma turbulence, which do not contain nonlinear terms worse than bilinear, can be reduced by Fourier transformation to a system of equations of the form of Eq. (7).

If the terms in c_{ijk} are negligible, we have a classic linear problem. For small values of the nonlinear term, a method of perturbation can probably render some services. In turbulence, the nonlinear terms become dominant, and a fundamentally different situation appears. In fact, the terms in b_{ij} either become negligible, or add only some minor complications to the general strategy. For simplicity, we shall now drop these linear terms. They could be carried along in a detailed treatment.

The coefficients c_{ijk} grow with the first indices. Indeed, the basic equations have terms such as $v_o(\partial v_o/\partial x_o)$ that lead to coefficients proportional to the wavenumbers. Thus the coefficients having large first indices will have large values, since they correspond to large wavenumbers.

It follows that a function such as F_{865} fluctuates faster than a function such as F_{32}; this introduces important statistical differences between the various functions.

The problem is to determine the statistical properties of the F_i, starting from the set of coefficients c_{ijk} generated by the equations of Navier-Stokes, and from some suitable assortment of initial conditions.

III. A MATHEMATICAL MODEL

In order to examine the reasoning of Kraichnan with maximum clarity, I shall occasionally refer to a model problem, which is not related to any particular flow but is simply defined by Eq. (7) with a convenient choice of the coefficients c_{ijk}.[1] For the model, these coefficients are simply taken from a collection of random numbers having a root mean square of unity and an average of zero. The initial conditions, that is, the values of all the F_i at $t = 0$, are also chosen at random.

If the number of functions F is less than 100 or so, the problem can be integrated numerically with sufficient accuracy at an acceptable cost. I have found that statistical reproducibility occurs already when N is larger than 20, and satisfactory numerical experiments can be carried out with $N = 48$. This is a great relief from turbulent cases and it is principally due to the fact that all functions F_i have similar behavior because all coefficients have the same order of magnitude.

The state of the flow, or that of the model, is specified at any time by the N functions F_i. This corresponds to a point in a space of N dimensions. The kinetic energy of the flow is defined as $E = \sum F_i F_i$, and it follows from Eq. (7) that E is a constant, since the viscous terms have been discarded. In the case of the model, it is easy to adjust the coefficients c_{ijk} so that E is also an invariant. Then, as the time increases, the representative point stays on the surface of the hypersphere, in the N-dimensional space.

Starting from some initial conditions at $t = 0$, we can imagine the trajectory, perhaps like the solid line shown in Fig. 1.

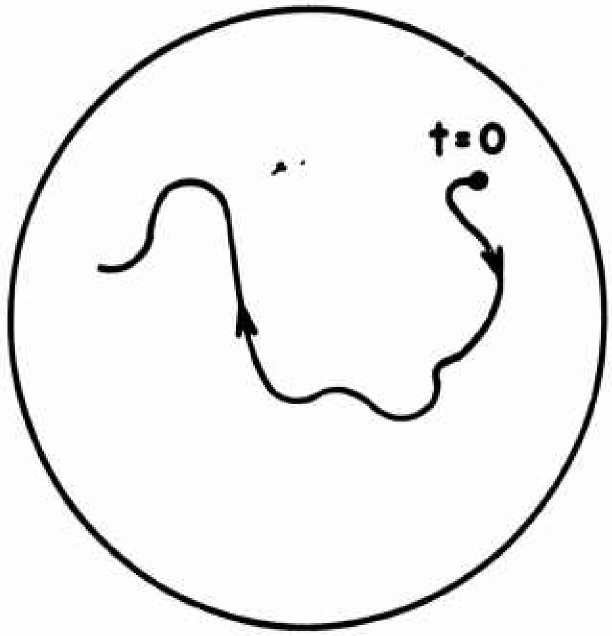


Figure 1. Solution of a Turbulent Energy Preserving Problem in a Multi-dimensional Space

In Fig. 2 we see the result of a numerical integration of the model with $N = 48$. A particular function F_1 displays random fluctuations. The other 47 functions have a statistically similar behavior.

Figure 2. Behavior of a Component in a Numerical Experiment

IV. THE AUTOCORRELATION FUNCTION

The physicist confronted with the signal of Fig. 2 immediately realizes the futility of a theory giving every wiggle of the solution. Some statistical reduction is necessary, and the first question is, "What is the power spectrum?" The fluid dynamicist will ask the equivalent question, "What is the auto-correlation?"

This function is defined as

$$H(\tau) = \overline{F(t)\,F(t+\tau)} \tag{8}$$

In the case of turbulence, this function varies with the index i and with the corresponding wavenumber. This function $H(k,\tau)$ could be measured, but this has not been done. Perhaps the results of Favre et al.[2] could furnish such information.

In the case of the model, H is independent of the index and the computer can easily determine $H(\tau)$. The results give the bell-shaped curve shown in Fig. 3.

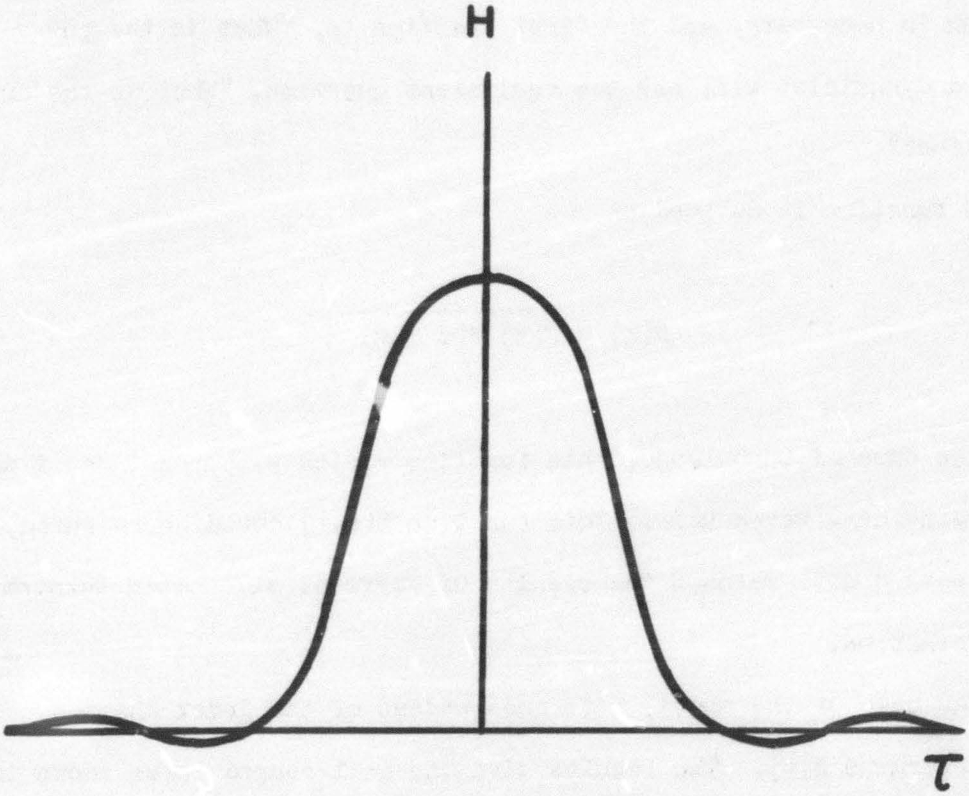


Figure 3. Autocorrelation Function of the Model Problem

V. EQUATION FOR THE AUTOCORRELATION FUNCTION

The theoretician is now challenged to present an expression for $H(\tau)$. His first step will be to form an equation for H. For this purpose we denote $F(t')$ as F'. After multiplying the basic Eq. (7) by F' and averaging, we obtain

$$-(dH/d\tau) = \sum F_i'(dF_i/dt) = \sum c_{ijk} \overline{F_i' F_j F_k} \qquad (9)$$

On the right hand side, we have a large number of terms, each one a triple correlation. Since the left hand side is of the order of magnitude of unity, in some appropriate units of time, each triple correlation must be fairly small. Thus the various components F_i are almost, but not quite, independent. The classical method would now lead to equations for the triple correlations, in terms of quadruple correlations, followed by some drastic assumptions on the quadruple correlations.[3] Kraichnan attempts to directly evaluate the triple correlations by means of a regression function. We shall now follow this new approach.

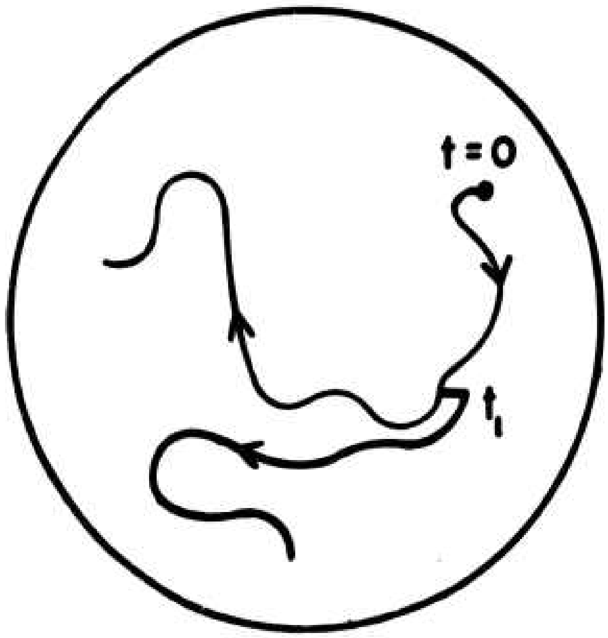


Figure 4. Effect of a Small Perturbation on a Turbulent Solution

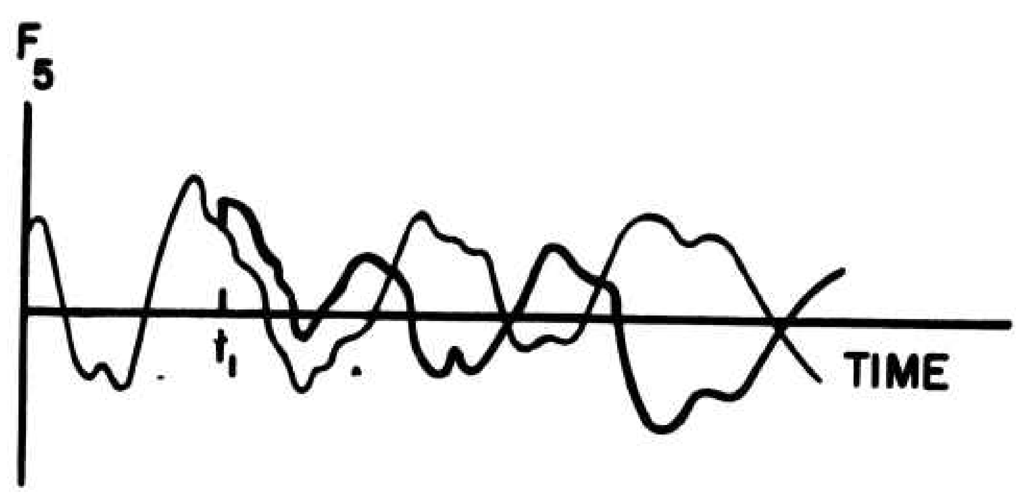


Figure 5. Effect of a Small Perturbation on a Component of the Numerical Model

VI. THE REGRESSION FUNCTION

Let us come back to the model and examine the stability of the solution. After obtaining the solution for a given set of coefficients and a given set of initial conditions, let us reset the computer and repeat the numerical integration. Since every element of the computer is fully controlled, the integration is completely reproducible. Even the errors (truncation, finite step size, etc.) occur in exactly the same fashion.

At time t_1 we can stop the computer and modify one of the 48 functions. Let us say that we increase F_5 by $\epsilon = 0.01$. (The root mean square of F_i is unity). The computer is then instructed to resume integration. The effect of the disturbance is illustrated in Fig. 4. At time t_1, the trajectory makes a sharp step in the direction of the fifth axis; thereafter it follows the undisturbed trajectory for a while, gradually drifting away.

After a long time, the perturbed solution is far away from the undisturbed solution, somewhere on the hypersphere. All correlation between the two trajectories has been lost. The two functions F_j and $F_{j,\epsilon}$ can be plotted, as in Fig. 5. The difference $\Delta f = F_{5,\epsilon} - F_5$ can be divided by ϵ, so that it jumps from zero to unity at $t = t_1$.

After a long time, this normalized difference becomes very large, of the order of $1/\epsilon$, but the probability that it will be positive is equal to the probability that it will be negative.

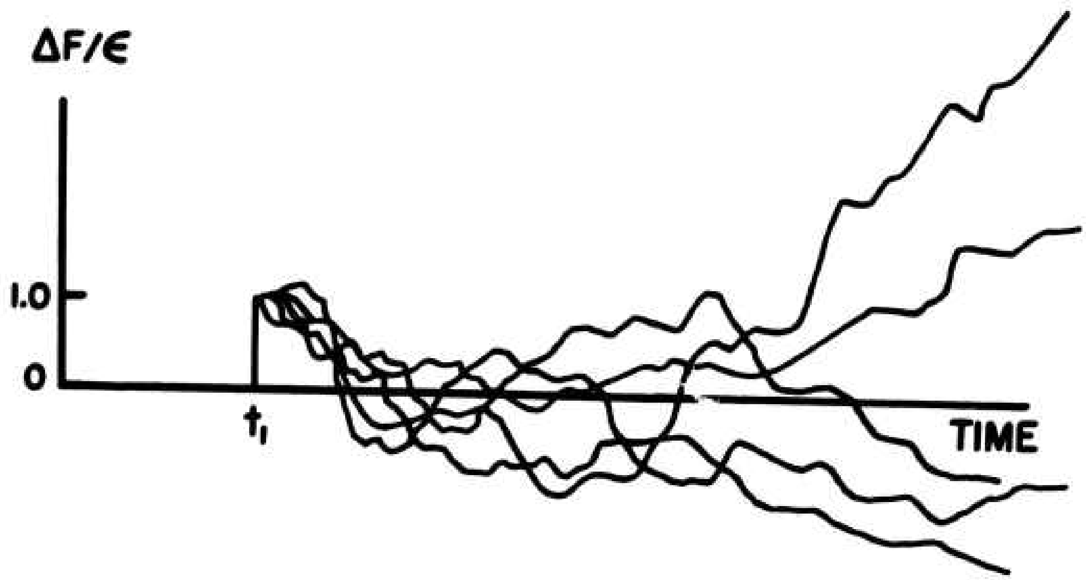


Figure 6. Normalized Effect of a Small Perturbation for Different Initial Conditions (Numerical Model)

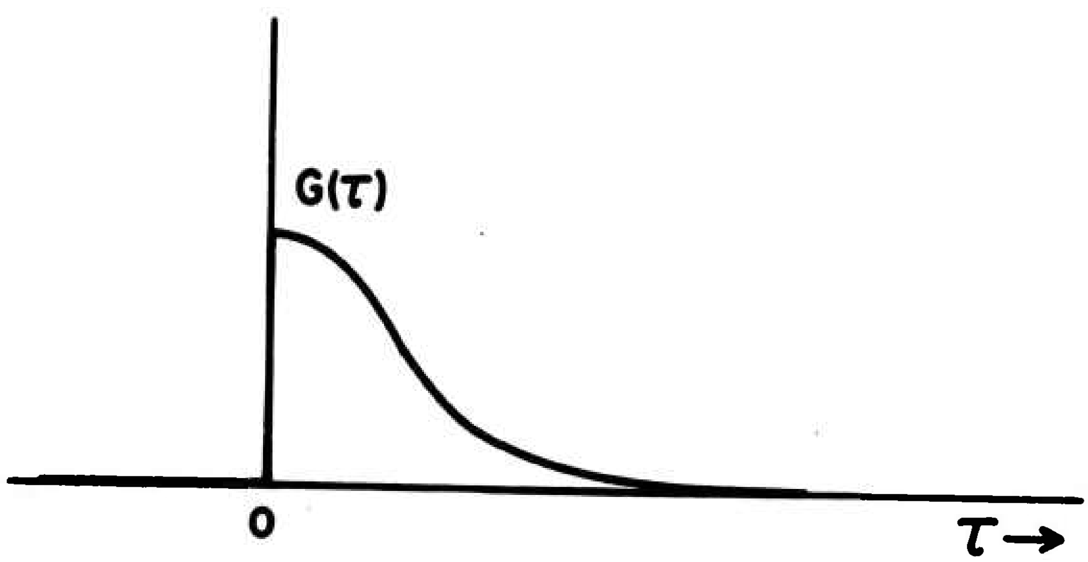


Figure 7. Regression Function of the Model Problem

Once again, the situation calls for a statistical reduction. Let us repeat the procedure, changing only the initial conditions at $t = 0$. For each set of initial conditions, we have another normalized difference, jumping from zero to unity at t_1 and eventually becoming very large.

A family of such results is shown in Fig. 6. Clearly, the behavior of the difference can be predicted for some time after t_1, but the long-range performance is completely uncertain.

The computer can be programmed to determine the average of the family of curves shown in Fig. 6. This average will be noted as $G(\tau)$ with $\tau = t - t'$. For $\tau<0$, $G = 0$ (see Fig. 7). This is the regression function, and it indicates the probable effect of a small pulse applied at $\tau = 0$. The energy of the pulse is gradually distributed among all the N functions F. Thus, G describes not a dissipation of energy, but a loss of information.

At $t = t_1$, the probability distribution of F_5 and of any other component is Gaussian. Immediately after the perturbation, the probability distribution of F_5 is a Gaussian, shifted by the amount ϵ. In time, this probability will return to normal; the function G describes this regression process.

In order to be sure that the individual differences indeed become very large, in absolute value, I have asked the computer to determine the function K defined as

$$K = \overline{\Delta F^2}^{\frac{1}{2}} / \epsilon \qquad (10)$$

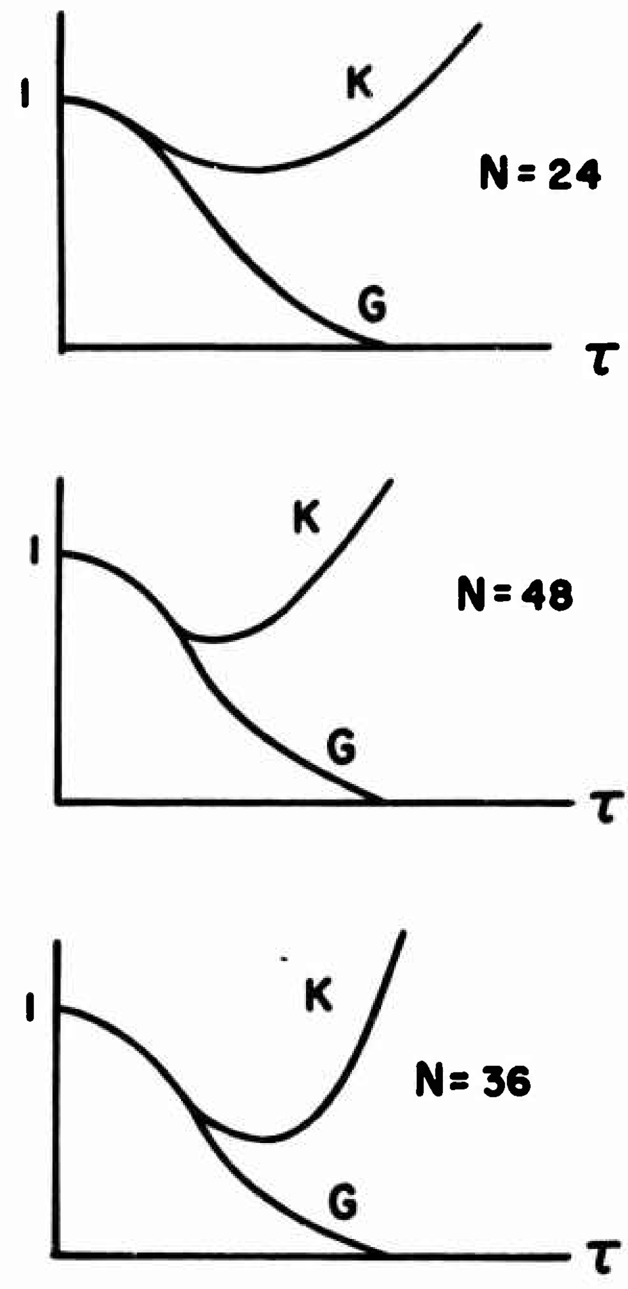


Figure 8. Functions G and K for Models of Increasing Intricacy

Some results are shown in Fig. 8, for various values of N. The horizontal scales are suitably changed so that the functions G all look alike. The functions K indeed grow and eventually level off at $1/e$. When G is almost zero, the growth of K is exponential.

As N increases, note that K follows G for a longer time before taking off. This means that, as the system becomes more intricate, the actual effect of a pulse can be predicted with increasing accuracy. Indeed, the gap between K and G is a measure of the statistical variations between differences ΔF.

For $N = 10^{23}$, K may practically vanish before the final ascension. Such systems would have a "hydrodynamical" response to perturbations!

VII. APPLICATION OF THE REGRESSION FUNCTION

Let us now suppose that the perturbation is not concentrated on one instant at time t_1, but is continuously applied to the system.

If $\psi(t)$ is a small fluctuating force, generated externally, we have

$$dF_5/dt = \sum c_{5jk} F_j F_k + \psi \tag{11}$$

If we had a linear system, the effect of the external force could be superposed on an ordinary solution, leading to a result such as

$$F_5 = \mathcal{F}(t) + \int_{-\infty}^{t} G(t-s)\, \psi(s)\, ds \tag{12}$$

where $\mathcal{F}$ denotes a solution in the absence of external force. For a linear system, G is simply the effect of a single pulse, without need for a statistical averaging or a limitation to small amplitudes.

With a nonlinear system of sufficient intricacy, we can perhaps write an equation of the form of Eq. 12, except that $\mathcal{F}$ should become some function closely related to a solution of the undisturbed equations.

Essentially, in an expression such as Eq. (12), the integral contains those effects of ψ which can be predicted by means of the regression function, while the term in $\mathcal{F}$ contains the effect of the initial conditions plus those consequences of the forcing function which cannot be predicted. Thus, Eq. (12) is not very useful if we look for a formal solution of the problem.

However, Eq. (12) becomes valuable when we form the correlation $\overline{\psi(t')F_5}$ between the forcing function and the solution of the nonlinear problem.

From Eq. (12) we obtain

$$\overline{\psi' F} = \overline{\psi' \mathcal{F}} + \int_{-\infty}^{t} G(t-s)\, \Psi(t-s)\, ds \tag{13}$$

where $\Psi = \overline{\psi'\psi}$ is the autocorrelation of the forcing function. The integral is a simple operation on two bell-shaped functions. The correlation $\overline{\psi'\mathcal{F}}$ is far more difficult to evaluate. In fact, it would be impossible to continue the theory if we could not dispose of this term! Essentially, since the predictable effects of ψ are given by the integral term, it is reasonable to assume that $\overline{\psi'\mathcal{F}}$ is negligible. We shall do so.

VIII. THE TRIPLE CORRELATIONS

The previous section can now be used to express the triple correlations. Consider, for example, the equation giving F_5 and assume that, among the many terms of the right hand side, we find one proportional to the product F_7F_2. What is the correlation $\overline{F_5F_7F_2}$? By pulling one term outside of the summation and indicating its presence as $\phi = c_{572}\, F_7F_2$ we have

$$dF_5/dt = \sum_{\text{ONE TERM LESS}} c_{5jk}F_jF_k + \phi \tag{14}$$

The argument of the previous section can be used, treating one term as a small perturbation applied in the presence of almost all the other interactions.

It leads to the following result:

$$\overline{F_7F_2F_5(t)} = c_{572}\int_{-\infty}^{t} G(t-s)\overline{F_7F_2\ F_7(s)\ F_2(s)}\, ds \tag{15}$$

This approximation is valid if the functions F_7,F_2 and F_5 are independent, except for the linkage by the coefficient c_{572}. Additional terms must be entered, proportional to similar effects caused by c_{725} and c_{257}. Such terms are labelled "direct interactions" because they involve only the three functions forming the triple correlation. There are also many possibilities for indirect interactions, involving other functions. For example, a term F_7F_{50} could contribute to the generation of F_{100} and a term F_2F_{50} could contribute to the generation of F_{300}. Now, if the product $F_{100}F_{300}$ appears in the equation for dF_5/dt, an approximate analysis shows that the product $F_2F_{50}F_7F_{50}$ affects F_5. Thus, through the action of F_{50}, we have another linkage.

Kraichnan assumes that the direct interactions are the dominant ones. This amounts to saying that the various Fourier components are correlated three-by-three, but not in more numerous groups.

IX. THE EQUATION FOR H

As shown in Eq. (15), a triple correlation can be expressed by an integral over a function G and a quadruple correlation. Let us use the drastic assumption of Milliontschikov to reduce the quadruple correlation to a product of two double correlations. Later on, in Section XII, we shall comment on this crucial step and examine the various kinds of quadruple correlations to offer some justification.

Triple correlations can now be given in terms of doubles. Thus the road started at Eq. (9) leads to an expression of the type:

$$dH/dt = c \ldots c \ldots \int_{-\infty}^{\tau} G(\tau - s)\, H(s)\, H(s)\, ds \tag{16}$$

This is the first of two master equations established by Kraichnan. If G is known, Eq. (16) gives H.

X. THE EQUATION FOR G

The application of a perturbation ϵ on the function F_5 produces the difference ΔF, seen in Fig. 6. It also causes a disturbance in every other component of the motion, and we can define N-1 differences δF_j. While ΔF is discontinuous at $t = t_1$, the quantities δF_j are continuous but each shows a discontinuity in some derivative. Thus, if F_9 is directly modified by F_5, the derivative of δF_9 will jump. If F_3 is directly affected by F_9 but not by F_5, the second derivative of δF_3 will jump, etc.

The averaged value of any function δF_j is zero $(j \neq 5)$, since the perturbations are "modulated" by various functions F_k.

Immediately after the perturbation, and during the decay of G, the difference ΔF_5 and the many other differences δF_j are small. Thus the basic equations can be linearized and we write:

$$d/dt \; \Delta F_5 = 2 \sum c_{5jk} \delta F_j F_k \tag{17}$$

$$d/dt \; \delta F_j = \sum c_{j5p} \Delta F_5 F_p + 2 \sum c_{jpq} \delta F_p F_q \tag{18}$$

In order to determine $G = \Delta F_5 / \epsilon$, we must eliminate the quantities δF_j. The equations for δF_j show that every one is a response of the system to a perturbation $\Delta F_5 \; F_p$. The predictable part of this response is

$$\delta F_j = \sum c_{j5p} \int_{-\infty}^{t} G(t-s)\ \Delta F_5(s)\ F_p(s)\ ds \tag{19}$$

After substituting Eq. (19) in Eq. (17) and averaging, we obtain the equation for G

$$dG/d\tau = 2 \sum c\ldots c\ldots \int_{-\infty}^{\tau} G(\tau-s)\ G(s)\ H(s)\ ds \tag{20}$$

This is the second master equation. Together with Eq. (16) it forms a closed system of equations that, in general, gives $H(k,\tau)$ and $G(k,\tau)$.

The theory is now complete. It remains to integrate a system of integro-differential equations giving a set of bell-shaped functions. This is easy for the model, but still a difficult task for any turbulent flow.

XI. THEORY FOR THE MODEL

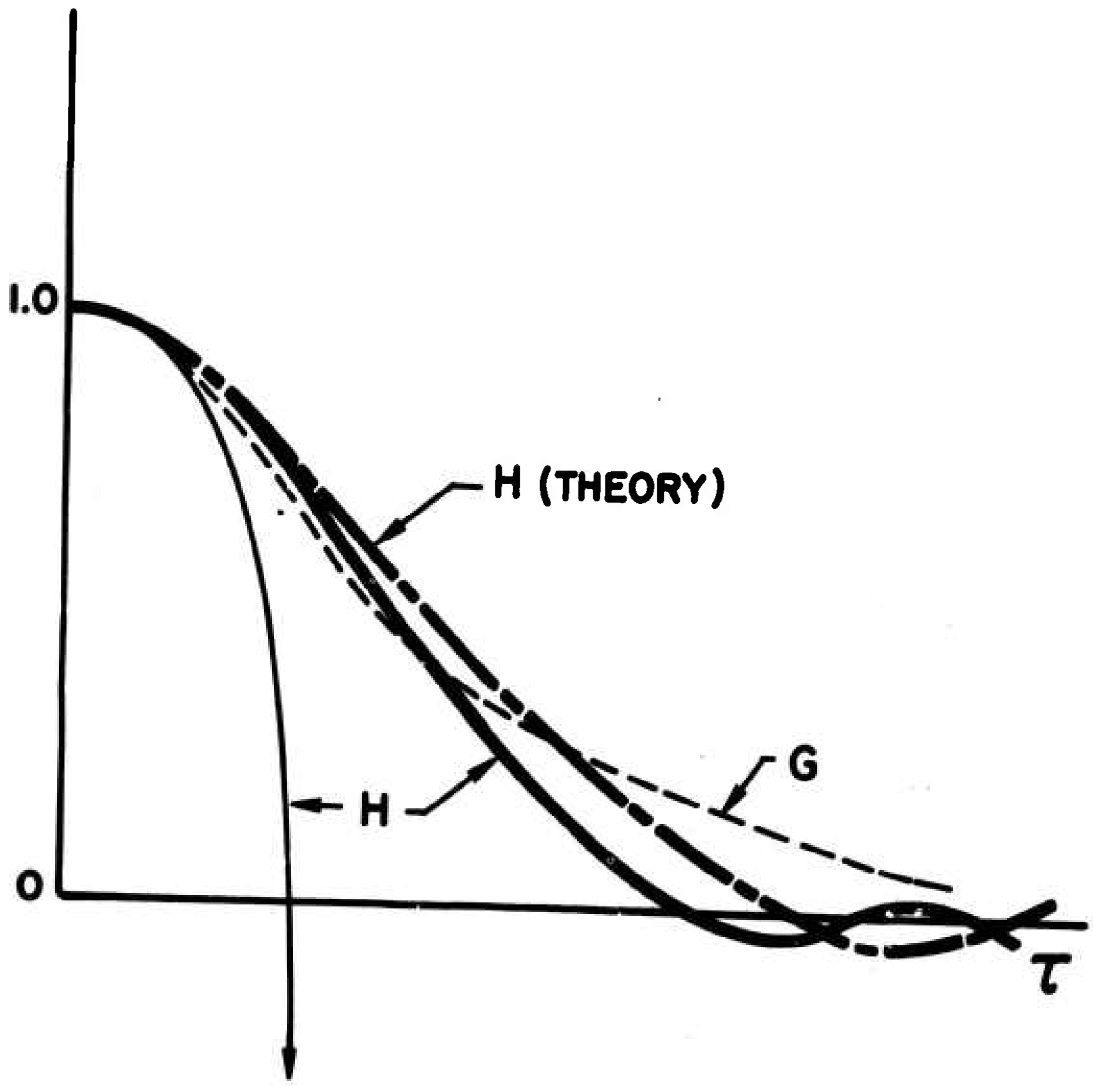
In the case of the mathematical model, the second master equation reduces to a form identical with the first equation, so that one finds $G = H$, provided that τ is positive. Thus the single equation is

$$dH/d\tau = -c\ldots c\ldots \int_0^{\tau} H(\tau-s)\, H^2(s)\, ds \tag{21}$$

It can be integrated easily by numerical techniques, and the results are shown in Fig. 9. The solid curve gives the experimental results, the dashes and dots show the solution of Eq. 21 from the theory of Kraichnan.

The dots show the experimental values for G. The discrepancies between theory and experiments are probably due to numerical errors in the integration of Eq. 21 or to limited averaging samples, or to the fact that the theory applies for $N = \infty$, while the experiments apply to $N = 48$.

In order to compare the new theory with older methods, we can form an equation for triple correlations in terms of quadruple correlations and use the drastic reduction to doubles. The result is shown in Fig. 9 by the thin curve, which plunges to $-\infty$. Thus, the new theory marks a clear progress.



Heavy solid line: From numerical experiments.
Heavy broken line: From Kraichnan's theory, Eq. (21).
Thin solid line: From discard of cumulants.
Thin broken line: Regression function, from numerical work.

Figure 9. Autocorrelation Functions

XII. ROLE OF THE QUADRUPLE CUMULANTS

In order to examine in detail the difference between old and new theories, we shall start by establishing an exact relation, a refreshing episode. Multiplying the basic Eq. (7) taken at time t by the same equation taken at time t' and averaging leads to

$$\overline{(dF_i'/dt)(dF_i/dt)} = \Sigma c_{ijk} \Sigma c_{ipq} \overline{F_j'F_k'F_pF_q} \tag{22}$$

If the process is statistically stationary in time, the first integration gives

$$dH/d\tau = -\sum\sum c_{ijk} c_{ipq} \int_0^\tau \overline{F_j'F_k'F_pF_q}\, ds \tag{23}$$

Note the double summation. The right hand side of Eq. (23) can be regarded as a product between a matrix $c_{ijk}\, c_{ipq}$ and a matrix formed by various quadruple correlations.

Let us now approximate these quadruple correlations by products of doubles. Since $\overline{F_i'F_j}$ vanishes unless $i = j$, the only terms of Eq. (23) that do not disappear are those found along certain diagonals of a matrix.

Thus the exact Eq. (23) reduces to an expression typical of these theories

$$dH/d\tau = -\sum c \ldots c \ldots \int_0^\tau HH\, ds \tag{24}$$

Note that the summation has been reduced to a single sum. If there is a small error in the evaluation of each quadruple correlation <u>along the diagonals</u>, it will have a small effect on the result. However, if there is a small error on each quadruple correlation <u>away from the diagonals</u>, the very large multitude of these terms may produce significant errors. Thus, it is the neglect of the cumulants <u>off</u> the diagonals that is likely to cause trouble.

The theory of Kraichnan leads to Eq. (16), which reduces to Eq. (24) if we take $G = 1$. Thus, the presence of the regression function in the final results is significant. Since G represents some effects of the complicated coupling between the modes, it perhaps accounts for the off-diagonal terms. How well is not known.

Finally, if we return to Eq. (16), we remember that Kraichnan uses the simplified form for the quadruple correlations at some stage of the argument. However, this reduction involves only special correlations along the diagonals and does not abridge a double sum to a single sum. Thus, it seems that the cumulants can be neglected only along the diagonals.

XIII. APPLICATION TO GRID TURBULENCE

The turbulence produced by a grid placed in a wind tunnel is generally characterized by a low Reynolds number. With a mesh M_o and a free stream U_o, we shall refer either to $R_M = U_o M_o/\nu$, of the order of 15,000,[4] or to $R_\lambda \approx (0.1 R_M)^{\frac{1}{2}}$, varying between 20 and 70.

The energy spectrum is too narrow to display the Kolmogoroff range. Kraichnan applied his theory in the case $R_\lambda = 40$, assuming a convenient form for the shape of the spectrum at $t = 0$. The turbulence decays, so that the functions H and G vary slowly with the time. The successive spectra are shown in Fig. 10. Note the decay of the energy and the production of small vortices. The decay proceeds according to a similarity law.

For this flow, Kraichnan determined the skewness factor S defined as

$$S = (\partial v_1/\partial x_1)^3 / \left[(\partial v_1/\partial x_1)^2\right]^{3/2} \tag{25}$$

where v_1 and x_1 are along the mean flow direction. Note that S is nondimensional. This factor plays an essential role in the study of nonlinear turbulent processes. It is related to short range triple correlations and it is also proportional to the rate of production of the mean square vorticity, and to a rate-of-deformation parameter.[5] Certain theories are based on an expression for the eddy viscosity,[6] which introduces one nondimensional universal constant

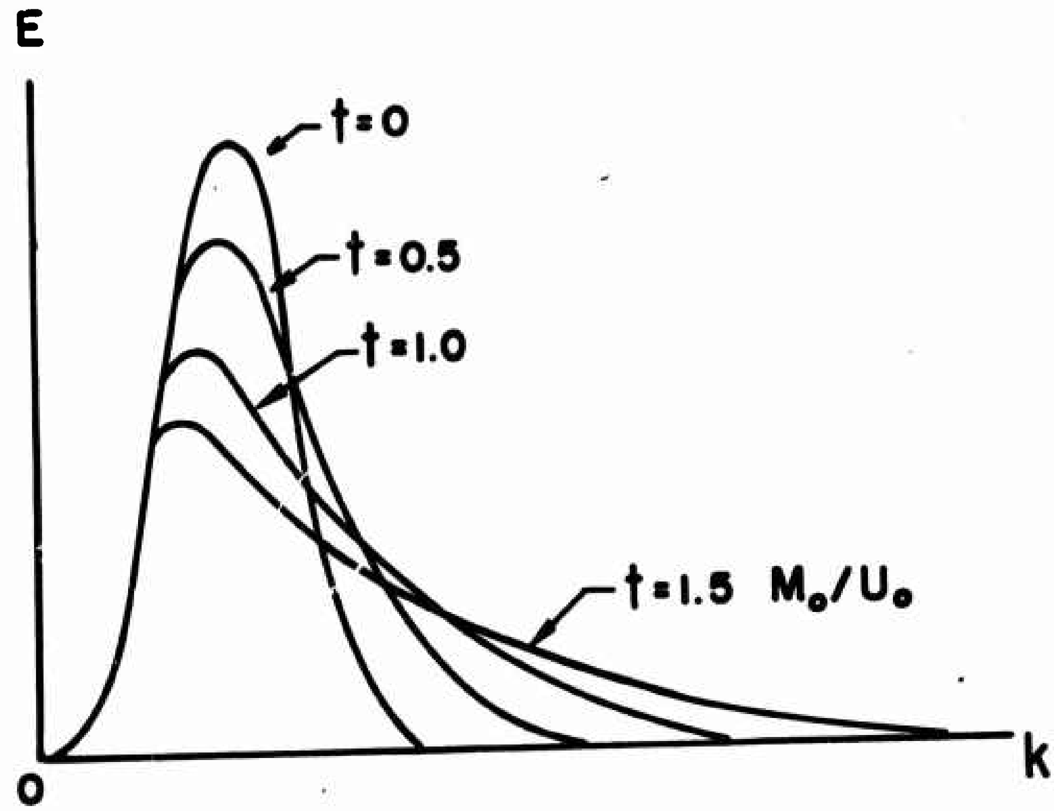


Figure 10. Decay of the Energy Spectrum
(According to Theory for R = 40)

that must be matched with experimental data. This constant is none but the factor S, in a special form.

In Kraichnan's treatment it is necessary, at $t = 0$, to specify the triple correlations; he started from provisions for zero triple correlations, which corresponds to $S = 0$ in Fig. 11. The rapid initial growth of S is compatible with previous work of Proudman and Reid,[7] which used the drastic treatment of quadruple correlations. However, the new theory finds that S rapidly levels off and retains the value $S = 0.4$ during the decay. This is in excellent agreement with the experiments of Townsend and others.

Note that Kraichnan does not adjust any constant in his determination of S. He also treats several similar cases, always with the limit $S \rightarrow 0.4$. A kinematic argument shows that S cannot exceed a value near 0.8.[5]

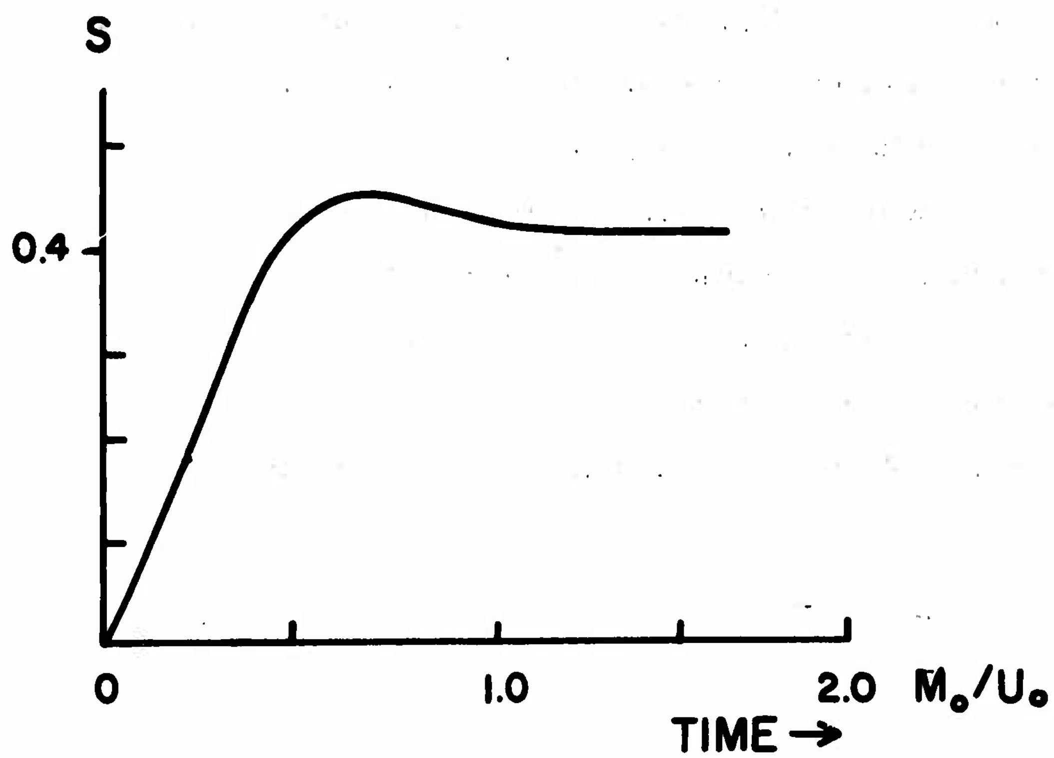


Figure 11. Skewness Factor
(According to Kraichnan for $R_\lambda = 40$. The Value Factor 0.4 Agrees with Many Experiments.)

XIV. THE KOLMOGOROFF SPECTRUM

In the atmosphere, the ocean, or a sufficiently large apparatus, the spectrum of turbulence shows a portion falling as $k^{-5/3}$. We shall consider the case of stationary turbulence, so that we must imagine some large scale mechanism furnishing the energy necessary to maintain the flow, despite the energy dissipation ϵ.[4]

In a first attempt to apply his theory, Kraichnan met with some difficulties. In the limit of very large Reynolds numbers (say $R_\lambda = 300$), the equation for $G(k,\tau)$ takes the form of a product

$$dG(k,\tau)/d\tau = \left\{\sum \Lambda\,(k',k'')\right\} \left\{ \int_{-\infty}^{\tau} \phi(k,\tau,s)\, ds \right\} \tag{26}$$

where the integration in s contains only a function of τ, k, and s. The sum, however, contains a contribution from each triangle in the wavenumber space such that $k' + k'' = k$. If either k' or k'' becomes smaller or comparable to the wave number K at which the spectrum is maximum, the behavior of Λ requires special attention. As a result, the final expression for the energy spectrum takes the form

$$E(k) = \epsilon^{2/3} \left[(a/k^{5/3}) + (b/K^{1/6} k^{3/2}) \right] \tag{27}$$

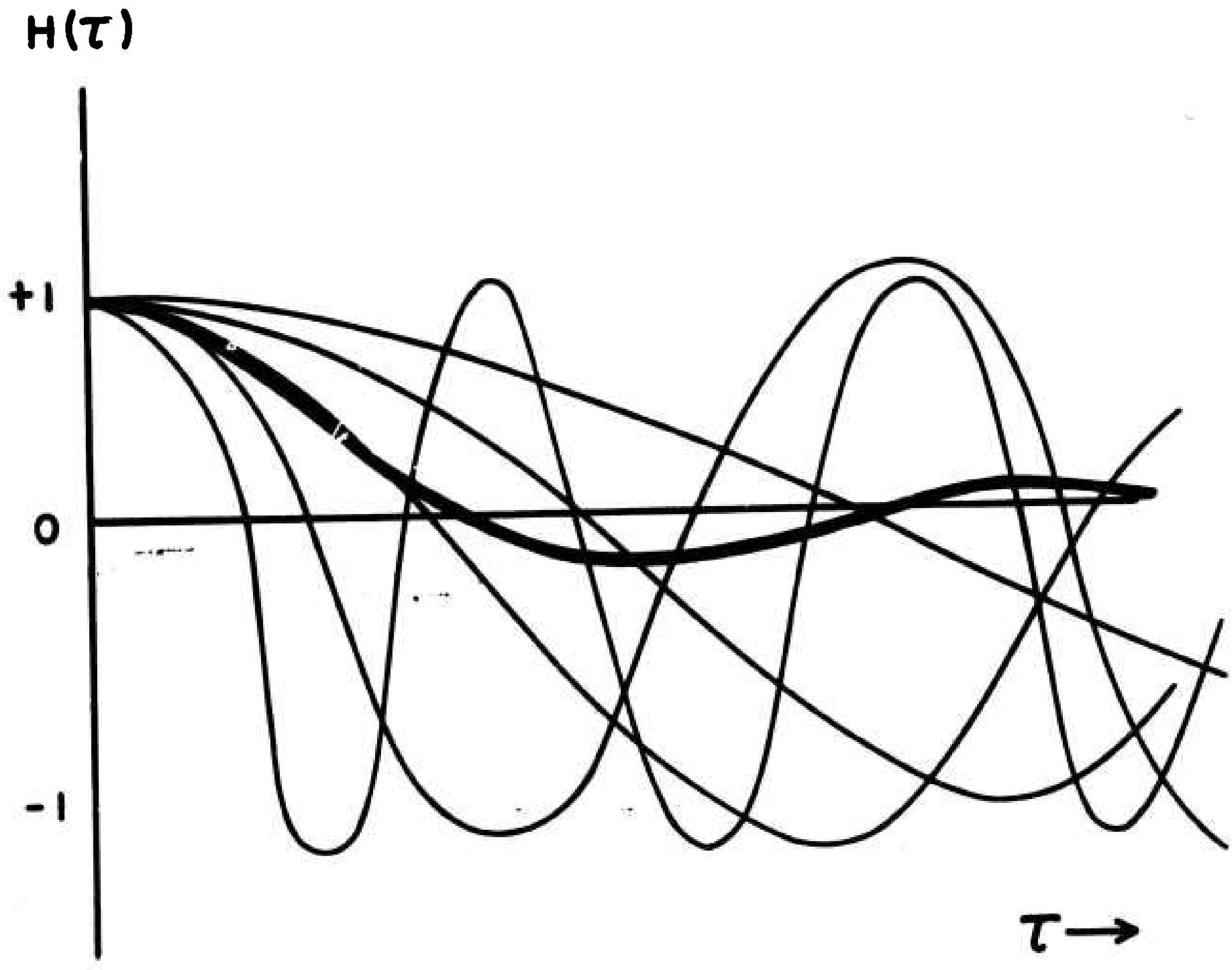


Figure 12. Autocorrelation Functions Produced by a Simple Shear Wave Transported at Various Uniform Speeds (The Averaged Effect is Shown by the Heavy Line)

The term proportional to a comes from values of k' and k" larger than K. The term proportional to b comes from the small values of k' or k", that is, from the large scale vortices.

It is clear that if k is large, the above expression falls as $k^{-3/2}$. If the singular behavior did not occur, b would vanish and the spectrum would fall as expected.

This situation prompted Kraichnan to a reexamination of the first form of his theory. He found that the argument that I have outlined in the previous pages suffers from a serious defect. It can be corrected if the correlations are defined in a sort of local frame of reference moving with the local fluid velocity. The papers dealing with the revised theory are easily identified: the title always contains the name of Lagrange. They are not easily read.

To give you an idea of the nature of the difficulty, I shall consider a situation so simple that there are only two modes. First let us imagine a fluid at rest, animated by a single stationary shear wave such that

$v(x,y,z,t) = \sin \alpha x.$

Clearly, the correlation $\overline{v(x,t)\ v(x,t+\tau)}$ is independent of τ since the shear wave is stationary. Let us now displace the fluid with a supplementary velocity U, which is constant in space and in time. We assume that the shear wave is transported with the velocity U.

Then the velocity observed at a fixed point varies in time and the time correlation will fluctuate. For a particular value of U, the correlation will be given by one of the thin lines shown in Fig. 12. If we change the magnitude of the velocity U, we get some other oscillating correlation function; only the period varies.

An average over an ensemble of different velocities U or over time, if U changes slowly, will yield the bell-shaped correlation function shown by the thick line in Fig. 12.

Thus, the transport of a single shear wave by an assortment of large scale motions can produce a certain kind of correlation. However, one should not expect that the theory of Kraichnan will succeed in this case. The theory must be applied only to those correlations that are not produced by random translations of the same shear wave.

It is too early to say that the modification proposed by Kraichnan is the only possible one, but it is not surprising that, once the treatment for low k' and k'' is revised, the spectrum falls as $k^{-5/3}$. The Kolmogoroff law is already in the Eulerian theory. It must also be noted that the Lagrangian theory gives a numerical value of the universal spectral constant $\acute{a}$ that is in good agreement with experiments.

Turbulence at high Reynolds numbers raises many interesting questions. What is the skewness? It has not yet been measured experimentally. One could also take the signal proportional to a velocity fluctuation and pass it through a simple low-pass filter. The skewness of the filtered velocity could be determined and perhaps compared with theoretical results.

XV. CRITICISM OF THE THEORY AND COMMENTS

So far, the theory has not predicted many remarkable properties of turbulent flows, except in the field of MHD turbulence, where the confirmation may take a long time.

In ordinary turbulence, measurements of $H(k,\tau)$ would be interesting, since a comparison would be possible. A special effort should be made, perhaps, to specify the odd correlations of order higher than three. If the direct interaction approximation is valid, the triple correlations should determine those of order five, seven, etc. This would explain the results of Frenkiel and Klebanoff.[8]

Some scientists have not fully accepted the cascade theory and Richardson's poetry. In electronics-engineering, the energy supplied at 60 cps is often converted to much higher frequencies, say megacycles, without any intermediate oscillations. The spectrum has essentially two separate peaks. If the high frequency signal is intensely modulated, the spectrum could fill the entire range. Thus, one can perhaps conceive other theories of turbulence, with direct links between the very large and the very small eddies. Intermittency might perhaps play an essential role. So far these are only speculations, and the theory of Kraichnan still stands as the best that we have.

REFERENCES

1. R. Betchov, "Introduction to the Kraichnan theory," Proc. Symp. Dynamics Fluids and Plasmas, College Park, Maryland, October 1965 (Academic Press, New York, to be published), eds. A. J. Faller and S. I. Pai.

2. A. J. Favre, J. J. Gaviglio, and R. J. Dumas, "Space-time double correlations and spectra in a turbulent boundary layer," J. Fluid Mech. 2 (4), 313-342 (June 1957); also "Further space-time correlations of velocity in a turbulent boundary layer," J. Fluid Mech. 3 (4), 344-356 (January 1958).

3. M. D. Millionshchikov, "On the theory of homogeneous isotropic turbulence," Compt. Rend. Acad. Sci. URSS 32, 615-635 (1941).

4. G. K. Batchelor, The Theory of Homogeneous Turbulence (Cambridge University Press, New York, 1953).

5. R. Betchov, "Inequality concerning production of vorticity in isotropic turbulence," J. Fluid Mech. 1 (5), 497-504 (November 1965).

6. W. Heisenberg, "On the statistical theory of turbulence," Z. Phys. 124 (7-12), 628-657 (1948).

7. I. Proudman and W. H. Reid, "On the decay of a normally distributed and homogeneous turbulent velocity field," Phil. Trans. Roy. Soc. London, Ser. A: 247, 163-189 (1954-1955).

8. F. N. Frenkiel and P. S. Klebanoff, "Higher-order asymmetries in turbulent flow," Compt. Rend. 260 (23), 6030-6032 (June 1965).

ANNOTATED BIBLIOGRAPHY

R. H. Kraichnan, "Lagrangian-History Closure Approximation for Turbulence," Phys. Fluids 8 (4), 575-598 (April 1965).

Difficult paper, leading to Kolmogoroff's spectrum.

__________, "Preliminary Calculation of the Kolmogorov Turbulence Spectrum," Phys. Fluids 8 (5), 995-997 (May 1965).

This short note leads to -5/3 law by use of Lagrangian effects.

__________, "Inertial-Range Spectrum of Hydromagnetic Turbulence," Phys. Fluids 8 (7), 1385-1387 (July 1965).

Note on MHD effects.

__________, "Approximations for Steady-State Isotropic Turbulence," Phys. Fluids 7 (8), 1163-1168 (August 1964).

Uses functions of wave number and frequency instead of wave number and time. Relates to theory proposed by Edwards.

__________, "Decay of Isotropic Turbulence in the Direct-Interaction Approximation," Phys. Fluids 7 (7), 1030-1048 (July 1964).

Gives numerical results for low Reynolds number (grid turbulence); spectra, skewness, similarity, dissipation, transfer. Also describes machine procedure.

See Erratum, 8 (1), 210 (January 1965).

__________, "Diagonalizing Approximation for Inhomogeneous Turbulence," Phys. Fluids 7 (8), 1169-1177 (August 1964).

Extension of theory to shear flows and thermally driven turbulence.

__________, "Direct-Interaction Approximation for Shear and Thermally-Driven Turbulence," Phys. Fluids 7 (7), 1048-1062 (July 1964).

See Erratum, 8 (3), 552 (March 1965).

__________, "Kolmogorov's Hypotheses and Eulerian Turbulence Theory," Phys. Fluids 7 (11), 1723-1734 (November 1964).

Argument that Eulerian approach is not suitable for deriving Kolmogorov's results.

__________, "Mixed Lagrangian-Eulerian Approach to Turbulent Dispersion," Phys. Fluids 7 (10), 1717-1719 (October 1964).

Short note.

__________, "Relation between Lagrangian and Eulerian Correlation Times of a Turbulent Velocity Field," Phys. Fluids 7 (1), 142-143 (January 1964).

Suggests that Lagrangian correlation times should be shorter than Eulerian correlation times.

__________, "Direct-Interaction Approximation for a System of Several Interacting Simple Shear Waves," Phys. Fluids 6, (11), 1603-1609 (November 1963).

A model shows that direct-interaction is better than quasi-normality, with three and five shear waves.

__________, "Turbulent Thermal Convection at Arbitrary Prandtl Number," Phys. Fluids 5 (11), 1374-1389.

__________ and E. A. Spiegel, "Model for Energy Transfer in Isotropic Turbulence," Phys. Fluids 5 (5), 583-588 (May 1962).

__________, "Dynamics of Nonlinear Stochastic Systems," J. Math. Phys. 2 (1), 124-148 (January-February 1961).

__________, "Comments on 'Space-Time Correlations in Stationary Isotropic Turbulence,'" Phys. Fluids 2 (3), 334 (May-June 1959).

__________, "The Structure of Isotropic Turbulence at Very High Reynolds Numbers," J. Fluid Mech. 5 (4), 497-543 (May 1959).

This is a detailed paper, containing essential theoretical results.

__________, "Higher Order Interactions in Homogeneous Turbulence Theory," Phys. Fluids 1 (4), 358-359 (July-August 1958).

A brief note on indirect interactions.

__________, "Irreversible Statistical Mechanics of Incompressible Hydromagnetic Turbulence," Phys. Rev. 109 (5), 1407-1422 (March 1, 1958).

Gives discussion of G equation.

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13. ABSTRACT

The principal assumptions used by R. Kraichnan in formulating a theory of turbulence are reviewed, and their meaning is examined in the light of a mathematical model. A regression function is defined and used to evaluate the triple correlations. The importance of the fourth order cumulants is discussed. The merits of the theory are illustrated by comparison with experimental results -- in the case of the mathematical model -- of grid turbulence (skewness factor) and of turbulence at very large Reynolds number.

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